

SCHUR PROPERTIES OVER SOME LIPSCHITZ-FREE SPACES

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ABSTRACT. In this paper we show that the Lipschitz-free space over some proper metric spaces (every closed ball is compact) linearly embed into a ℓ_1 -sum of finite dimensional subspaces of itself. We also prove that under natural conditions, the Lipschitz-free space over a proper metric space has the 1-Schur property and the 1-strong Schur property. Then we turn to the study of the classical Schur property giving conditions on a metric space to ensure that its Lipschitz-free space has this property. We finish with the study of those properties on the Lipschitz-free space over the metric space originate from a p -Banach space ($0 < p < 1$).

1. INTRODUCTION

Let (M, d) be a pointed metric space and let 0 be its distinguished point. Consider $Lip_0(M)$ the set of all real valued Lipschitz functions f defined on M and verifying $f(0) = 0$. This space equipped with the norm

$$\|f\|_L := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} ; x \neq y \in M \right\}$$

is a Banach space (we refer to [21] for the study of this space). For $x \in M$, define the evaluation functional $\delta_x \in Lip_0(M)^*$ by $\delta_x(f) = f(x)$ for every $f \in Lip_0(M)$. It is not difficult to see that $\mathcal{F}(M) := \overline{\text{span} \{\delta_x ; x \in M\}}^{\|\cdot\|}$ is a predual of $Lip_0(M)$. This last space is called the Lipschitz-free space over M , we will refer to it as the free space over M for convenience. This terminology is due to Godefroy and Kalton, it first appeared in [5]. We refer to this last paper for basic properties of free spaces. However, this space was studied before, for example in [21] where Weaver called it the Arens-Eels space over M .

A fundamental property of free spaces is that every Lipschitz map between two metric spaces $L: M_1 \rightarrow M_2$ admits a linearization $\hat{L}: \mathcal{F}(M_1) \rightarrow \mathcal{F}(M_2)$ such that $\|\hat{L}\| = Lip(L)$ ($Lip(L)$ being the Lipschitz constant of L) and such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{L} & M_2 \\ \delta_1 \downarrow & \searrow & \downarrow \delta_2 \\ \mathcal{F}(M_1) & \xrightarrow{\hat{L}} & \mathcal{F}(M_2) \end{array}$$

where δ_i is an isometry defined by $\delta_i: x \in M_i \mapsto \delta_i(x) \in \mathcal{F}(M_i)$. This property permits in a particular way to transform a Lipschitz problem into a linear problem. Thus, although free spaces are easy to define, their structure is difficult to analyze. For example we know that $\mathcal{F}(\mathbb{N})$ is linearly isometric to $l_1(\mathbb{N})$ and $\mathcal{F}(\mathbb{R})$ is linearly isometric to $L_1(\mathbb{R})$. But the structure

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of $\mathcal{F}(\mathbb{R}^2)$ is poorly understood. We know that $\mathcal{F}(\mathbb{R}^2)$ does not embed linearly into L_1 (see [20]), $\mathcal{F}(\mathbb{R}^2)$ has the metric approximation property (see [5]), and $\mathcal{F}(\mathbb{R}^2)$ is weakly sequentially complete (i.e weakly Cauchy sequence are weakly convergent). More generally Cúth, Doucha and Wojtaszczyk proved in [2] that $\mathcal{F}(M)$ is weakly sequentially complete for every $M \subseteq \mathbb{R}^n$, and it is also proved in the same paper that the free space over any infinite metric space admits a complemented subspace linearly isomorphic to ℓ_1 .

Free spaces are related to old open problems. An important question in non linear classification of Banach spaces is to determine whether a Lipschitz equivalence between separable Banach spaces implies that these two spaces are linearly isomorphic. For example we do not know if a separable Banach space that is Lipschitz isomorphic to ℓ_1 is also linearly isomorphic to ℓ_1 . In order to provide a positive answer to that question, it is sufficient to prove that $\mathcal{F}(\ell_1)$ is complemented in its bidual (which also an open question). Another classical example is the following question (see the remark after Proposition 4.4 in [15]): Does the free space over a uniformly discrete metric space has the bounded approximation property ? A negative answer would provide an equivalent norm on ℓ_1 that fails the metric approximation property, and thus would solve a 50-years old problem.

It should now be clear for the reader that exploring the structure of free spaces is a big challenge. Here is a list (absolutely non exhaustive) of recent results, some of which will further interest us. In [3], A. Dalet proved that the free space over a countable proper metric space (i.e every closed ball is compact) or over an ultrametric proper metric space is a dual space with the metric approximation property. In [4] Godard characterizes free spaces that are linearly isometric to a subspace of an L_1 -space as free spaces over a subset of an \mathbb{R} -tree. Finally, in [9] Hájek, Lancien and Pernecká proved that the free space over a proper countable metric space has the Schur property. In this note we shall improve this last result.

2. QUANTITATIVE SCHUR PROPERTIES ON LIPSCHITZ-FREE SPACES OVER SOME PROPER METRIC SPACES

In the following, we will only consider real Banach spaces. For completeness, we recall here some definitions.

Definition 2.1. Let (M, d) be a metric space. We define the two following closed subspaces of $Lip_0(M)$:

$$\begin{aligned} lip_0(M) &:= \left\{ f \in Lip_0(M) : \lim_{\varepsilon \rightarrow 0} \sup_{0 < d(x,y) < \varepsilon} \frac{|f(x) - f(y)|}{d(x,y)} = 0 \right\}, \\ S_0(M) &:= \left\{ f \in lip_0(M) : \lim_{r \rightarrow \infty} \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)} = 0 \right\}. \end{aligned}$$

The first space $lip_0(M)$ is called the little Lipschitz space over M . This terminology first appeared in [21] where they were considered on compact metric spaces. We will say that a subspace $S \subset Lip_0(M)$ *separates points uniformly* if there exists a constant $c \geq 1$ such that for every $x, y \in M$ and every $\varepsilon > 0$ there is $f \in S$ satisfying $\|f\| \leq c + \varepsilon$ and $f(x) - f(y) = d(x, y)$. Weaver showed in [21] (Theorem 3.3.3) that if M is a compact metric space then $lip_0(M)$

separates points uniformly if and only if it is an isometric predual of $\mathcal{F}(M)$. More generally, the same result holds for proper metric spaces (i.e every closed ball is compact), more precisely if M is a proper metric space then $S(M)$ uniformly separates points if, and only if, it is an isometric predual of $\mathcal{F}(M)$ ([3]). Let us recall that a subspace Z of X^* , where X is a Banach space, is called C -norming (with $C \geq 1$) if for every $x \in X$, $\|x\| \leq C \sup_{z^* \in B_Z} |z^*(x)|$. It is well known that $\text{lip}_0(M)$ or $S_0(M)$ is C -norming if and only if it separates points uniformly with constant $c = C$ in the previous definition (see Proposition 3.4 in [15]).

In order to state the main result of this part we need to assume that the considered Banach space has the Metric Approximation Property (denoted (MAP)). For the sake of completeness we recall here a definition (among many characterizations) of this approximation property. We say that a Banach space X has the Approximation Property (AP) if for every $\varepsilon > 0$, and every finite sequence of points $x_1, \dots, x_n \in X$, there exist a finite rank operator $T \in \mathcal{B}(X)$ such that $\|Tx - x\| \leq \varepsilon$. Let $\lambda \geq 1$, if in the above definition T can always be chosen so that $\|T\| \leq \lambda$, then we say that X has the λ -Bounded Approximation Property (λ -(BAP)). When X has the 1-(BAP) we say that X has (MAP).

We now turn to the definition of the Schur property and some quantitative version of it.

Definition 2.2. Let X be a Banach space. We say that X has the Schur property if every weakly null sequence $(x_n)_{n \in \mathbb{N}}$ in X is also $\|\cdot\|$ -convergent to 0.

Using Rosenthal's ℓ_1 theorem (see Theorem 10.2.1 page 252 in [1] for example), it is not difficult to prove the following characterization:

Proposition 2.3. *A Banach space X has the Schur property if and only if for every $\delta > 0$, every δ -separated sequence $(x_n)_{n \in \mathbb{N}}$ in the unit ball of X contains a subsequence that is equivalent to the unit vector basis of ℓ_1 , that is there exists a constant $K > 0$ (which may depend on the sequence considered) and a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that*

$$\sum_{i=1}^n |a_i| \geq \left\| \sum_{i=1}^n a_i x_{n_i} \right\| \geq \frac{1}{K} \sum_{i=1}^n |a_i|, \text{ for every } (a_i)_{i=1}^n \in \mathbb{R}^n.$$

In this case, we say that the sequence $(x_{n_k})_{k \in \mathbb{N}}$ is K -equivalent to the unit vector basis of ℓ_1 .

We will now define two quantitative versions of the Schur property. The first one is called the strong Schur property, it has been introduced for the first time by Johnson and Odell in [10]. We also refer to [6] for the 1-strong Schur property.

Definition 2.4. Let X be a Banach space.

- (1) We say that X has the strong Schur property if there exists a constant $K > 0$ such that, for every $\delta > 0$, any δ -separated sequence $(x_n)_{n \in \mathbb{N}}$ in the unit ball of X contains a subsequence that is $\frac{K}{\delta}$ -equivalent to the unit vector basis of ℓ_1 .
- (2) We say that X has the 1-strong Schur property if for every $\delta > 0$ and every $\varepsilon > 0$, any normalized δ -separated sequence $(x_n)_{n \in \mathbb{N}}$ in X contains a subsequence that is $(\frac{2}{\delta} + \varepsilon)$ -equivalent to the unit vector basis of ℓ_1 .

It is clear with the previous characterization that the strong Schur property implies the Schur property. And the Schur property is strictly weaker than the strong Schur property (see [22] or [11] for example).

We refer to [13] (Proposition 2.1) for some equivalent formulations of the strong Schur property.

Examples 2.5.

- (1) In [17] (Proposition 4.1), Knaust and Odell proved that if X has the property (S) and does not contain any isomorphic copy of ℓ_1 , then X^* has the strong Schur property. In particular ℓ_1 and all its subspaces have the strong Schur property. A Banach space has the property (S) if every normalized weakly null sequence contains a subsequence equivalent to the unit vector basis of c_0 , this is equivalent to the hereditary Dunford-Pettis property. We recall that a Banach space X has the Dunford-Pettis property if every continuous weakly compact operator from X into another Banach space Y transforms weakly compact sets in X into norm-compact sets in Y . And X has the hereditary Dunford-Pettis if all its closed subspaces have the Dunford-Pettis property.
- (2) In [6] (Lemma 3.4), Godefroy, Kalton and Li proved that a subspace of L_1 has the strong Schur property if and only if its unit ball is relatively compact in the topology of convergence in measure.

Before giving the second quantitative version of the Schur property, we need two moduli defined on sequences in a Banach space.

Definition 2.6. Let X be a Banach space, and let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in X . We write $\text{clust}_{X^{**}}(x_n)$ for the set of all *weak** cluster points of $(x_n)_{n \in \mathbb{N}}$ in X^{**} . Then we define the two following moduli:

- (1) $\delta(x_n) := \text{diam}\{\text{clust}_{X^{**}}(x_n)\}.$
- (2) $\text{ca}(x_n) := \inf_{n \in \mathbb{N}} \text{diam}\{x_k; k \geq n\}.$

The first moduli measures how far is the sequence from being weakly Cauchy and the second one measures how far is the sequence from being $\|\cdot\|$ -Cauchy.

We can now give the second quantitative version of the Schur property which has been introduced by Kalenda and Spurný in [11] more recently than the previous version.

Definition 2.7. Let X be a Banach space and let $C > 0$ be a constant. We say that X has the C -Schur property if for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in X :

$$\text{ca}(x_n) \leq C \delta(x_n)$$

In [11] the authors proved that the 1-Schur property implies the 1-strong Schur property, and that the 1-strong Schur property implies the 5-Schur property. To the best of our knowledge, the question whether the 1-strong Schur property implies the 1-Schur property is open.

We now state the main result of this paper:

Proposition 2.8. *Let (M, d) be a proper metric space such that $S_0(M)$ separates points uniformly and such that $\mathcal{F}(M)$ has (MAP). Then for any $\varepsilon > 0$, there exists a sequence $(E_n)_n$ of finite-dimensional subspaces of $\mathcal{F}(M)$ such that $\mathcal{F}(M)$ is $(1 + \varepsilon)$ -isomorphic to a subspace E of $(\oplus_n E_n)_{\ell_1}$.*

Proof. The first ingredient is a lemma of Godefroy, Kalton and Li (Lemma 3.1 in [6]):

Lemma. *Let V be a subspace of $c_0(\mathbb{N})$ with (MAP). Then for any $\varepsilon > 0$, there exists a sequence $(E_n)_n$ of finite-dimensional subspaces of V^* and a weak*-to-weak* continuous linear map $T: V^* \rightarrow (\oplus_n E_n)_{\ell_1}$ such that for all $x^* \in V^*$*

$$(1 - \varepsilon)\|x^*\| \leq \|Tx^*\| \leq (1 + \varepsilon)\|x^*\|.$$

The second ingredient is the following one. Generalizing a proof of Kalton in the compact case (Theorem 6.6 in [15]), Dalet has proved in [3] (Lemma 3.9) that if M is proper then the space $S_0(M)$ is $(1 + \varepsilon)$ -isomorphic to a subspace Z of $c_0(\mathbb{N})$.

Let us consider (M, d) a metric space satisfying the hypothesis of Proposition 2.8 and let us take $\varepsilon > 0$ arbitrary. Fix ε' such that $(1 + \varepsilon')^3 < 1 + \varepsilon$. First remark that $S_0(M)$ has (MAP) because this property passes to predual (see [8] or [19]). Next there exists a subspace Z of $c_0(\mathbb{N})$ such that $S_0(M)$ is $(1 + \varepsilon')$ -isomorphic to Z . Then note that Z has also the metric approximation property. Indeed Z^* is $(1 + \varepsilon)$ -isomorphic to $\mathcal{F}(M)$, so Z^* has the $(1 + \varepsilon)$ -bounded approximation property. Next, we get that Z has (MAP) from another theorem of Grothendieck (see again [8] or Theorem 1.e.15 in [19]): a separable dual space which has (AP) also has the metric approximation property. Consequently Z^* has (MAP) and thus Z has also (MAP). Thus we can apply the previous Lemma to Z so that there exists a sequence $(F_n)_n$ of finite-dimensional subspaces of Z^* such that Z^* is $(1 + \varepsilon')$ -isomorphic to a subspace F of $(\oplus_n F_n)_{\ell_1}$. Now $\mathcal{F}(M)$ is $(1 + \varepsilon')$ -isomorphic to Z^* so there exists a sequence $(E_n)_n$ of finite-dimensional subspaces of $\mathcal{F}(M)$ such that $(\oplus_n E_n)_{\ell_1}$ is $(1 + \varepsilon')$ -isomorphic to $(\oplus_n F_n)_{\ell_1}$. Then there exists E subspace of $(\oplus_n E_n)_{\ell_1}$ which is $(1 + \varepsilon')$ -isomorphic to F . It is easy to check that $\mathcal{F}(M)$ is $(1 + \varepsilon')^3$ -isomorphic to E , this completes the proof. \square

As a direct corollary of the previous result we get the following.

Corollary 2.9. *Let (M, d) as in Proposition 2.8. Then $\mathcal{F}(M)$ has the 1-Schur property (and thus also the 1-strong Schur property).*

Proof. We consider the same notation that we used in the previous proof. Since every E_n is of finite dimension, we can adapt the proof of the fact that ℓ_1 is 1-Schur in [11] (Theorem 3.1) in order to obtain that $(\oplus_n E_n)_{\ell_1}$ is 1-Schur. The 1-Schur property is stable passing to subspaces, so the subspace E has the 1-Schur property. We easily deduce from the fact that $\mathcal{F}(M)$ is $(1 + \varepsilon)$ -isomorphic to E that $\mathcal{F}(M)$ has the $(1 + \varepsilon)$ -Schur property.

To summarize, we proved that for every $\varepsilon > 0$, $\mathcal{F}(M)$ has the $(1 + \varepsilon)$ -Schur property. It is straightforward to check that this implies that $\mathcal{F}(M)$ has the 1-Schur property and thus also has the 1-strong Schur property (since 1-Schur implies 1-strong Schur). \square

Remark 2.10. In [12] it is proved (Theorem 1.1) that if X is a subspace of $c_0(\Gamma)$, then X^* has the 1-Schur property. So one can use this last result to prove Corollary 2.9 in a more general setting, that is:

Proposition 2.11. *Let M be a proper metric space such that $\text{lip}_0(M)$ is 1-norming, then $\mathcal{F}(M)$ has the 1-Schur property.*

But Proposition 2.8 gives more information on the structure of $\mathcal{F}(M)$ when this one has (MAP).

We now give some examples where Proposition 2.8 applies.

Examples 2.12.

- (1) (M, d) proper countable metric space (Theorem 2.1 and Theorem 2.6 in [3]).
- (2) (M, d) proper ultrametric metric space (Theorem 3.5 and Theorem 3.8 in [3]).
- (3) (M, d) compact metric space satisfying the assumptions of Proposition 6 in [7], for example the middle-third Cantor set.

3. LIPSCHITZ-FREE SPACES OVER THE METRIC SPACE ORIGINATE FROM A QUASI-BANACH SPACE

We know from Kalton's work (Theorem 4.6 in [15]) that if (M, d) is a metric space and ω is a nontrivial gauge then the space $\mathcal{F}(M, \omega \circ d)$ has the Schur property. But a careful reading of the proof reveals that the key ingredient is actually the fact that $\text{lip}_0(M, \omega \circ d)$ is always 1-norming (Proposition 3.5 in [15]). This lead us to the following result, we include the proof here for completeness even if the proof is very similar to Kalton's one.

Proposition 3.1. *Let (M, d) be a metric space such that $\text{lip}_0(M)$ is 1-norming. Then the space $\mathcal{F}(M)$ has the Schur property.*

Proof. According to Proposition 4.3 in [15], for every $\varepsilon > 0$, $\mathcal{F}(M)$ is $(1 + \varepsilon)$ -isomorphic to a subspace of $(\sum_{n \in \mathbb{Z}} \mathcal{F}(M_k))_{\ell_1}$ where M_k denotes the ball $\overline{B}(0, 2^k)$ centered at 0 and of radius 2^k . Moreover the Schur property is stable under isomorphism and passing to subspaces. So it suffices to prove the result under the assumption that M has finite radius.

Let us consider $(\gamma_n)_n$ a normalized weakly null sequence in $\mathcal{F}(M)$. We will show that

$$(1) \quad \forall \gamma \in \mathcal{F}(M), \liminf_{n \rightarrow +\infty} \|\gamma + \gamma_n\| \geq \|\gamma\| + \frac{1}{2},$$

from which we deduce that for every $\varepsilon > 0$, $(\gamma_n)_n$ admits a subsequence $(\frac{2}{\delta} + \varepsilon)$ -equivalent to the ℓ_1 -basis (see the end of the proof of Proposition 4.6 in [15]).

Fix $\varepsilon > 0$ and $\gamma \in \mathcal{F}(M)$. We can assume that γ is of finite support. Pick $f \in \text{lip}_0(M)$ with $\|f\|_L = 1$ and $\langle f, \gamma \rangle > \|\gamma\| - \varepsilon$. Next pick $\Theta > 0$ so that if $d(x, y) \leq \Theta$ then $|f(x) - f(y)| < \varepsilon d(x, y)$. Choose $\delta < \frac{\varepsilon \Theta}{2(1+\varepsilon)}$. Then by Lemma 4.5 in [15] we have

$$\inf_{|E| < \infty} \sup_n \text{dist}(\gamma_n, \mathcal{F}([E]_\delta)) = 0,$$

where $[E]_\delta = \{x \in M : d(x, E) \leq \delta\}$. Thus there exist a finite set $E \subset M$ such that E contains the support of γ and such that for each n we can find $\mu_n \in \mathcal{F}([E]_\delta)$ with $\|\gamma_n - \mu_n\| < \varepsilon$. Remark that $\mathcal{F}(E)$ is a finite dimensional space so that $\liminf_{n \rightarrow +\infty} \text{dist}(\gamma_n, \mathcal{F}(E)) \geq \frac{1}{2}$. Then by Hahn-Banach theorem, for every n there exist $f_n \in \text{lip}_0(M)$ verifying $\|f_n\|_L \leq 1 + \varepsilon$, $f_n(E) = 0$ and $\liminf_{n \rightarrow +\infty} \langle f_n, \gamma_n \rangle \geq \frac{1}{2}$. Now we define $g_n = (f + f_n)|_{[E]_\delta}$, then $g_n \in \text{lip}_0([E]_\delta)$ and we will show that $\|g_n\|_L < 1 + \varepsilon$. We will distinguish two cases to show this last property. First suppose that x and y are such that $d(x, y) < \Theta$, then

$$|g_n(x) - g_n(y)| \leq |f(x) - f(y)| + |f_n(x) - f_n(y)| \leq \varepsilon d(x, y) + d(x, y) = (1 + \varepsilon)d(x, y).$$

Second if x and y are such that $d(x, y) > \Theta$, then there exists $u, v \in E$ with $d(x, u) \leq \delta$ and $d(y, v) \leq \delta$, so that

$$\begin{aligned} |g_n(x) - g_n(y)| &\leq |f(x) - f(y)| + |f_n(x)| + |f_n(y)| \\ &= |f(x) - f(y)| + |f_n(x) - f_n(u)| + |f_n(y) - f_n(v)| \\ &\leq d(x, y) + 2(1 + \varepsilon)\delta \leq d(x, y) + \varepsilon\Theta \leq (1 + \varepsilon)d(x, y). \end{aligned}$$

We extend those functions g_n to M with the same Lipschitz constant and we still denote those extensions g_n for convenience. We now estimate the desired quantities.

$$\|\gamma + \mu_n\| \geq \frac{1}{1 + \varepsilon} \langle g_n, \gamma + \mu_n \rangle = \frac{1}{1 + \varepsilon} (\langle f, \gamma \rangle + \langle f, \mu_n \rangle + \langle f_n, \gamma \rangle + \langle f_n, \mu_n \rangle),$$

where

- $\langle f, \gamma \rangle > \|\gamma\| - \varepsilon$.
- $\limsup_{n \rightarrow \infty} |\langle f, \mu_n \rangle| \leq \varepsilon$, since $(\gamma_n)_n$ is weakly null and $\|\gamma_n - \mu_n\| < \varepsilon$.
- $\langle f_n, \gamma \rangle = 0$, since $\gamma \in \mathcal{F}(E)$.
- $\liminf_{n \rightarrow \infty} \langle f_n, \mu_n \rangle \geq \frac{1}{2} - \varepsilon$, since $\liminf_{n \rightarrow \infty} \langle f_n, \gamma_n \rangle > \frac{1}{2}$.

Thus,

$$\liminf_{n \rightarrow \infty} \|\gamma + \gamma_n\| \geq \frac{1}{1 + \varepsilon} (\|\gamma\| + \frac{1}{2} - 3\varepsilon) - \varepsilon.$$

This proves (1) since ε is arbitrary. □

The following proposition gives conditions on a metric space M to have $\text{lip}_0(M)$ 1-norming.

Proposition 3.2. *Let (M, d) be a metric space. Assume that for every $x \neq y \in M$ and $\varepsilon > 0$, there exists $N \subseteq M$ and a $(1 + \varepsilon)$ -Lipschitz map $T: M \rightarrow N$ such that $\text{lip}_0(N)$ is 1-norming for $\mathcal{F}(N)$, $d(Tx, x) \leq \varepsilon$ and $d(Ty, y) \leq \varepsilon$. Then $\text{lip}_0(M)$ is 1-norming.*

Proof. Let $x \neq y \in M$ and $\varepsilon > 0$. By our assumptions there exists $N \subseteq M$ and a $(1 + \varepsilon)$ -Lipschitz map $T: M \rightarrow N$ such that $\text{lip}_0(N)$ is 1-norming, $d(Tx, x) \leq \varepsilon$ and $d(Ty, y) \leq \varepsilon$. Since $\text{lip}_0(N)$ is 1-norming there exist $f \in \text{lip}_0(N)$ verifying $\|f\|_L \leq 1 + \varepsilon$ and $|f(Tx) - f(Ty)| = d(Tx, Ty)$. Now we define $g = f \circ T$ on M . By composition g is $(1 + \varepsilon)^2$ -Lipschitz and $g \in \text{lip}_0(M)$. Then a direct computation shows that g does the work.

$$\begin{aligned} |g(x) - g(y)| &= |f(Tx) - f(Ty)| = d(Tx, Ty) \\ &\geq d(x, y) - d(x, Tx) - d(y, Ty) \\ &\geq d(x, y) - 2\varepsilon. \end{aligned}$$

This ends the proof. □

For background on quasi-Banach spaces we refer the reader to [14, 16]. Let us recall first that a quasi-norm $\|\cdot\|$ on a real vector space X is an homogeneous map $X \rightarrow [0, \infty)$ such that $\|x\| = 0$ if and only if $x = 0$, and such that there exists k verifying $\|x + y\| \leq k(\|x\| + \|y\|)$ for every $(x, y) \in X^2$. The least k with this last property is often referred to as the modulus of concavity of the quasi-norm, we denote it k_X . The topology associated with a quasi norm

(which is metrizable as a Δ -norm, see the first chapter in [16]) is locally bounded. We say that X is a quasi-Banach space if X is complete for this topology.

Conversely on any Hausdorff topological vector space which is locally bounded, one can define a quasi-norm by using the Minkowski function of B a bounded neighborhood of the origin: $\mu(x) = \inf\{\lambda : x \in \lambda B\}$. Moreover if we assume that B is absolutely p -convex for some $0 < p \leq 1$ then the quasi-norm μ is p -subadditive, that is $\mu(x+y)^p \leq \mu(x)^p + \mu(y)^p$.

Now going back with $(X, \|\cdot\|)$ a quasi-normed space, we have the following basic and important theorem of Aoki-Rolewicz which can be interpreted as saying that if $0 < p \leq 1$ is given by $k_X = 2^{\frac{1}{p}-1}$, then for any $(x_k)_{k=1}^n \subset X$ we have

$$\left\| \sum_{k=1}^n x_k \right\|^p \leq 4 \left(\sum_{k=1}^n \|x_k\|^p \right).$$

It is then possible to replace $\|\cdot\|$ by an equivalent quasi-norm $|||\cdot|||$ which is p -subadditive. In fact we have $(1/4^{\frac{1}{p}})\|x\| \leq |||x||| \leq \|x\|$ if $|||\cdot|||$ is defined by the formula

$$|||x||| = \inf \left\{ \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} : \sum_{i=1}^n x_i = x \right\}.$$

We say that X is a p -normed space if the quasi-norm on X is p -subadditive, and then the topology on X is induced by the metric defined by $d(x, y) = \|x - y\|^p$. A quasi-Banach space with an associated p -norm is also called a p -Banach space.

Let $0 < p < 1$ and consider X_p a p -Banach space. We denote $M_p = (X_p, d_p)$ the metric space where the metric is the p -norm of X_p to the power p : $d_p(x, y) = \|x - y\|_p^p$. As mentioned before, we know that $lip_0(M, \omega \circ d)$ is always 1-norming when ω is a nontrivial gauge ω . But in our case we consider a quasi-norm composed with the nontrivial gauge $\omega(t) = t^p$. Thus, we can expect to have the same result. We will see that we need to use more arguments to overpass the difference that there exists between a norm and a quasi-norm. However, techniques that are employed here are inspired from Kalton's ones.

We begin with a very basic Lemma.

Lemma 3.3. *Let $0 < p < 1$ and $n \in \mathbb{N}$. Then we have the following inequalities*

$$\|x\|_1 \leq \|x\|_p \leq n^{\frac{1-p}{p}} \|x\|_1,$$

for every $x \in \mathbb{R}^n$.

Proof. The inequality $\|x\|_1 \leq \|x\|_p$ is obvious and a simple application of Holder's inequality gives $\|x\|_p \leq n^{\frac{1-p}{p}} \|x\|_1$. \square

From now on, M_p^n denote (ℓ_p^n, d_p) and we use the notation $\|\cdot\|$ for the p -norm of the p -Banach ℓ_p^n . In order to prove our first result about the structure of $\mathcal{F}(M_p^n)$, we need the following technical Lemma.

Lemma 3.4. *Let $R > 0$, $0 < p < 1$ and $n \in \mathbb{N}$. Then, there exist a Lipschitz function $\varphi: M_p^n \rightarrow M_p^n$ such that φ is the identity map on $\overline{B}(0, R)$, is null on $M_p^n \setminus B(0, 2R)$ and the Lipschitz constant of φ does not depend on R .*

Proof. Let us define $A = \overline{B}(0, R) \cup (M_p^n \setminus B(0, 2R)) \subset M_p^n$ (balls are considered for d_p) and $\phi: (A, d_p) \rightarrow M_p^n$ such that ϕ is the identity on $\overline{B}(0, R)$ and is null on $M_p^n \setminus B(0, 2R)$. It is easy to check that ϕ is 1-Lipschitz. We now write $\phi = (\phi_1, \dots, \phi_n)$. Then for every k $\phi_k: (A, d_p) \rightarrow (\mathbb{R}, |\cdot|^p)$ is 1-Lipschitz, thus $\phi_k: (A, \|\cdot\|) \rightarrow (\mathbb{R}, |\cdot|)$ is also 1 Lipschitz. Now the right side of the inequality in Lemma 3.3 implies that $\phi_k: (A, \|\cdot\|_1) \rightarrow (\mathbb{R}, |\cdot|)$ is $n^{\frac{1-p}{p}}$ -Lipschitz. So we can extend each ϕ_k without increase the Lipschitz constant and we denote φ_k those corresponding extensions. Summarizing we have $\varphi_k: (\mathbb{R}^n, \|\cdot\|_1) \rightarrow (\mathbb{R}, |\cdot|)$ which is $n^{\frac{1-p}{p}}$ Lipschitz and $\varphi_k|_A = \phi_k$. Now the right side of the inequality in Lemma 3.3 implies that $\varphi_k: \ell_p^n \rightarrow (\mathbb{R}, |\cdot|)$ is $n^{\frac{1-p}{p}}$ Lipschitz. So $\varphi_k: M_p^n \rightarrow (\mathbb{R}, |\cdot|^p)$ is n^{1-p} Lipschitz. To conclude, just remark that $\varphi = (\varphi_1, \dots, \varphi_n): M_p^n \rightarrow M_p^n$ is n^{2-p} -Lipschitz and verifies the desired properties. \square

We are now able to prove the following result.

Proposition 3.5. *If $0 < p < 1$ and $n \in \mathbb{N}$ then the space $\mathcal{F}(M_p^n)$ is isometric to $S_0(M_p^n)^*$ and so in particular $S_0(M_p^n)$ is 1-norming.*

Proof. In order to prove this result, we will first prove that $S_0(M_p^n)$ is C_n -norming for some $C_n > 0$. And then we will deduce the desired result using a duality argument (that is using a result of Dalet that we mentioned at the beginning of section 2, see [3]).

For every $m \in \mathbb{N}$ and $t \geq 0$, we define the following function

$$\omega_m(t) = \inf\{s^p + m(t - s) : 0 \leq s \leq t\}$$

which is continuous, non-decreasing and subadditive. Note that $\lim_{m \rightarrow +\infty} \omega_m(t) = t^p$.

Let $x \neq y \in M_p^n$. Since $(\ell_1^n)^* \equiv \ell_\infty^n$, by Hahn-Banach theorem there exist $x^* \in \ell_\infty^n$ such that $\|x\|_\infty = 1$ and $\langle x^*, x - y \rangle = \|x - y\|_1$. According to Lemma 3.3 this gives $\langle x^*, x - y \rangle \geq n^{\frac{p-1}{p}} \|x - y\|_p$. From now on we denote $F := n^{\frac{1-p}{p}} x^*$ and we see F as an element of $(\ell_p^n)^* \equiv \ell_\infty^n$ of norm $\|F\|_{(\ell_p^n)^*} \leq n^{\frac{1-p}{p}}$.

Let us consider $R > 2 \max(\|x\|^p, \|y\|^p)$ and $\varphi: M_p^n \rightarrow M_p^n$ given by the Lemma 3.4 (we denote C its Lipschitz constant). Of course we can see φ as a $C^{\frac{1}{p}}$ -Lipschitz function from ℓ_p^n to ℓ_p^n . We then consider f_m defined on M_p^n by

$$f_m(z) = \omega_m(|F(\varphi(z)) - F(y)|) - \omega_m(|F(y)|).$$

We will prove that our functions f_m belongs to $S_0(M_p^n)$ and do the job. Thus for $z \neq z' \in M_p^n$ we compute

$$\begin{aligned} |f_m(z) - f_m(z')| &= |\omega_m(|F(\varphi(z)) - F(y)|) - \omega_m(|F(\varphi(z')) - F(y)|)| \\ &\leq \omega_m(|F(\varphi(z)) - F(\varphi(z'))|) \\ &= \omega_m(|F(\varphi(z) - \varphi(z'))|). \end{aligned}$$

Using “ $s = t$ ” in the definition of ω_m we have

$$|f_m(z) - f_m(z')| \leq |F(\varphi(z) - \varphi(z'))|^p \leq n^{1-p} d_p(\varphi(z), \varphi(z')) \leq C n^{1-p} d_p(z, z').$$

Thus, f_m is d_p -Lipschitz with $\|f_m\|_L \leq Cn^{1-p}$. Now using “ $s = 0$ ” in the definition of ω_m we get

$$|f_m(z) - f_m(z')| \leq m|F(\varphi(z) - \varphi(z'))| \leq mn^{\frac{1-p}{p}} \|\varphi(z) - \varphi(z')\| \leq C^{\frac{1}{p}} mn^{\frac{1-p}{p}} \|z - z'\|.$$

This provides the fact that $f_m \in \text{lip}_0(M_p^n)$. It remains to prove that f_m satisfies the flatness condition at infinity to get $f_m \in S_0(M_p^n)$. To this aim, fix $\varepsilon > 0$ and pick $k > 2$ such that $\frac{2Cn^{1-p}}{(k-2)} \leq \varepsilon$. Now let z and z' be in M , and let us discuss by cases:

- If z and z' are not in $\overline{B}(0, kR)$, then $|f_m(z) - f_m(z')| = 0 < \varepsilon$.
- Now suppose that $z \notin \overline{B}(0, kR)$ and $z' \in \overline{B}(0, kR)$. Now we can still distinguish two more cases:
 - First assume that $z' \notin \overline{B}(0, 2R)$. Then again $|f_m(z) - f_m(z')| = 0 < \varepsilon$.
 - On the other hand, if $z' \in \overline{B}(0, 2R)$, then

$$\begin{aligned} \frac{|f_m(z) - f_m(z')|}{d_p(z, z')} &\leq \frac{|F(\varphi(z'))|^p}{(k-2)R} \\ &\leq \frac{n^{1-p} \|\varphi(z')\|^p}{(k-2)R} \\ &\leq \frac{Cn^{1-p}(2R)}{(k-2)R} \leq \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves that $f_m \in S_0(M)$. To end the first part of the proof just notice now that

$$|f_m(x) - f_m(y)| = \omega_m(|F(x) - F(y)|) \geq \omega_m(d_p(x, y)) \xrightarrow{m \rightarrow +\infty} d_p(x, y).$$

Thus $S_0(M_p^n)$ is Cn^{1-p} -norming.

We are now moving to the duality argument. Remark that M_p^n is a proper metric space, so using a result of Dalet that we mentioned at the beginning of section 2 (see [3]), we have that $S_0(M_p^n)^* \equiv \mathcal{F}(M_p^n)$. Thus, obviously $S_0(M_p^n)$ is 1-norming. \square

Of course this last result still holds for every metric space originate from a p -Banach space X_p of finite dimension.

Corollary 3.6. *Let $0 < p < 1$ and $n \in \mathbb{N}$. We consider M_p the metric space originate from a p -Banach space X_p which is of finite dimension. Then the space $\mathcal{F}(M_p)$ is isometric to $S_0(M_p)^*$ so in particular $S_0(M_p)$ is 1-norming.*

Proof. Note that since X_p is of finite dimension, it is isomorphic to ℓ_p^n for some $n \in \mathbb{N}$. Thus there is a bi-Lipschitz map between M_p and M_p^n , let us say $L: M_p \rightarrow M_p^n$ is bi-Lipschitz with $C_1 d_{M_p}(x, y) \leq d_{M_p^n}(L(x), L(y)) \leq C_2 d_{M_p}(x, y)$. Now we deduce easily that $S_0(M_p)$ is $\frac{C_2}{C_1}$ -norming. Indeed pick $x \neq y \in M_p$ and $\varepsilon > 0$. Since $S_0(M_p^n)$ is 1-norming there exist $f \in S_0(M_p^n)$ with Lipschitz constant less than $1 + \varepsilon$ such that

$$|f(L(x)) - f(L(y))| = d_{M_p^n}(L(x), L(y)) \geq C_1 d_{M_p}(x, y).$$

Now $f \circ L: M_p \rightarrow \mathbb{R}$ is Lipschitz with Lipschitz constant less than $C_2(1 + \varepsilon)$. Moreover as the composition of a bi-Lipschitz map with an element of $S_0(M_p^n)$ we know that $f \circ L \in S_0(M_p)$. Thus $S_0(M_p)$ is $\frac{C_2}{C_1}$ -norming. Since M_p is proper, now by what becomes a routine argument $\mathcal{F}(M_p)$ is isometric to $S_0(M_p)^*$ and thus $S_0(M_p)$ is 1-norming. \square

Corollary 3.7. *Let $0 < p < 1$ and $n \in \mathbb{N}$. We consider M_p the metric space originate from a p -Banach space X_p which is of finite dimension. Then there exists a sequence $(E_n)_n$ of finite-dimensional subspaces of $\mathcal{F}(M_p)$ such that $\mathcal{F}(M_p)$ is $(1 + \varepsilon)$ -isomorphic to a subspace E of $(\oplus_n E_n)_{\ell_1}$.*

Proof. The aim is to show that all assumptions of Proposition 2.8 are satisfied for $\mathcal{F}(M_p)$. According to Corollary 3.6 $S_0(M_p)$ is 1-norming. Now in [18] (Corollary 2.2) the authors show that if M is a doubling metric space (that is there exist $D(M) \geq 1$ such that any ball $B(x, r)$ can be covered by $D(M)$ open balls of radius $R/2$) then $\mathcal{F}(M)$ has the $C(M)$ -BAP where the constant $C(M)$ depends on the dimension and on the doubling-constant $D(M)$. And we also know that if $M \subset \mathbb{R}^n$ then M is doubling. Thus in our case $\mathcal{F}(M_p)$ has the $C(M_p)$ -BAP. Since it is a dual space, we get from a result of Grothendieck (see [8] or Theorem 1.e.15 in [19]) that $\mathcal{F}(M_p)$ actually has the (MAP). Thus all the assumptions of Proposition 2.8 are satisfied. \square

We now turn to the study of the structure of $\mathcal{F}(M_p)$ with more general assumptions on M_p . In particular we now pass to infinite dimensional spaces and the aim is to explore the behavior of $\mathcal{F}(M_p)$ regarding properties such as the (MAP) and the Schur property. To do so, we will assume that X_p is a p -Banach space which admits a monotone Finite Dimensional Decomposition (shortened in monotone FDD). In particular a space which admits a monotone Schauder basis such as ℓ_p satisfies this assumption. We start with the study of the Schur property. Using our Proposition 3.2 we manage to prove the following result.

Proposition 3.8. *Let $0 < p < 1$ and X_p be a p -Banach space which admits a monotone FDD. Then $\text{lip}_0(M_p)$ is 1-norming and $\mathcal{F}(M_p)$ has the Schur property.*

Proof. The aim is to applied Proposition 3.2. So we just have to show that M_p satisfies the corresponding hypothesis. Since X_p admits a monotone FDD there exist a sequence $(X_k)_{k \in \mathbb{N}}$ of finite dimensional subspaces of X_p such that every $x \in X$ admits a unique representation of the form $x = \sum_k x_k$ with $x_k \in X_k$. And if we denote P_n the projections from X_p to X_n defined by $P_n(x) = \sum_{k=1}^n x_k$ then $\sup_n \|P_n\| = 1$. Notice that those projections are actually 1-Lipschitz from M_p to $M_{p,n}$ where $M_{p,n} = (X_n, d_p)$.

Fix $x \neq y \in M_p$ and $\varepsilon > 0$. We can write $x = \sum_k x_k$, $y = \sum_k y_k$ with $x_k, y_k \in X_k$ for every k . Now fix $N \in \mathbb{N}$ such that $d_p(x, P_N(x)) < \varepsilon$ and $d_p(y, P_N(y)) < \varepsilon$. Since each X_k is of finite dimension, the space $(\sum_{k=1}^N X_k, \|\cdot\|)$ is of finite dimension and thus by corollary 3.6 $S_0(A)$ is 1 norming where $A = (\sum_{k=1}^N M_{p,k}, d_p)$. So in particular $\text{lip}_0(A)$ is 1-norming. Thus according to lemma 3.2 $\text{lip}_0(M_p)$ is 1-norming and so $\mathcal{F}(M_p)$ has the Schur property by Proposition 3.1. \square

Remark 3.9. We know from Kalton's work, that if (M, d) is a metric space and ω is a non-trivial gauge then $\text{lip}_0(M, \omega \circ d)$ is a 1-norming subspace of $\text{Lip}_0(M, \omega \circ d)$. So in order to prove Proposition 3.8 one can think in writing our distance from M_p as the composition of a gauge and another distance. But except in dimension 1 it is never possible to do so. Indeed, it is sufficient to prove that it is not the case in M_p^2 . We argue by contradiction

and so we assume that there exists ω a nontrivial gauge and d a distance on \mathbb{R}^2 such that $\|x - y\|_p^p = \omega(d(x, y))$ for every $x \neq y$. Now we consider the points $x = (t_x, 0)$, $y = (0, t_y)$. Straightforward computations shows that $\|x - y\|_p^p = |t_x|^p + |t_y|^p = \|x\|_p^p + \|y\|_p^p$. Since d is a distance and ω is a gauge we have

$$\begin{aligned} |t_x|^p + |t_y|^p &= \omega(d(x, y)) \leq \omega(d(x, 0) + d(y, 0)) \\ &\leq \omega(d(x, 0)) + \omega(d(y, 0)) = |t_x|^p + |t_y|^p. \end{aligned}$$

Thus $\omega(d(x, 0) + d(y, 0)) = \omega(d(x, 0)) + \omega(d(y, 0))$. From where we deduce that ω is additive, and so is such that $\omega(t) = t\omega(1) = t$. This contradicts the fact that ω is a nontrivial gauge.

We finish here by proving our last result about the (MAP). We keep the same notation than in Proposition 3.8.

Proposition 3.10. *Let $0 < p < 1$ and X_p be a p -Banach space which admits a monotone FDD. Then $\mathcal{F}(M_p)$ has the (MAP).*

Proof. Let $\mu_1, \dots, \mu_n \in \mathcal{F}(M_p)$ and $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ and $\nu_1, \dots, \nu_n \in \mathcal{F}(A)$ (where $A = (\sum_{k=1}^N M_{p,k}, d_p)$) such that $\|\mu_k - \nu_k\| \leq \varepsilon$. Since $\mathcal{F}(A)$ has (MAP) according to the proof of Corollary 3.7, there exist $T: \mathcal{F}(A) \rightarrow \mathcal{F}(A)$ a finite rank operator such that $\|T\| \leq 1$ and $\|T\nu_k - \nu_k\| \leq \varepsilon$ for every k . Since $P_N: M_p \rightarrow A$ is a 1-Lipschitz retraction, the linearization $\hat{P}_N: \mathcal{F}(M_p) \rightarrow \mathcal{F}(A)$ is projection of norm 1. This lead us now to consider the operator $\hat{P}_n \circ T: \mathcal{F}(M_p) \rightarrow \mathcal{F}(M_p)$ for which direct computations shows that it does the work. Indeed for every k :

$$\begin{aligned} \|\hat{P}_n \circ T\mu_k - \mu_k\| &\leq \|\hat{P}_n \circ T\mu_k - \hat{P}_n \circ T\nu_k\| + \|\hat{P}_n \circ T\nu_k - \nu_k\| + \|\mu_k - \nu_k\| \\ &\leq \|\hat{P}_n \circ T\| \|\mu_k - \nu_k\| + \|T\nu_k - \nu_k\| + \varepsilon \\ &\leq 3\varepsilon. \end{aligned}$$

□

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